

The Sunada construction and the simple length spectrum

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Abstract

We show that certain families of iso-length spectral hyperbolic surfaces obtained via the Sunada construction are not generally simple iso-length spectral.

1 Introduction

Let M be a compact Riemannian manifold. The **length spectrum** $L(M)$ of M is the set of all lengths of closed geodesics on M counted with multiplicities. Two manifolds M_1 and M_2 are said to be **iso-length spectral** if $L(M_1) = L(M_2)$.

In [10], Sunada provided a method to construct iso-length spectral manifolds that are frequently not isometric (see also [4, Ch.11-13]). This requires a notion from group theory.

Let G be a finite group. Two subgroups H and K of G are said to be **almost conjugate** if, for any $g \in G$,

$$|H \cap (g)| = |K \cap (g)|,$$

where (g) denotes the conjugacy class of g in G .

Theorem (Sunada). *Let M_0 be a closed Riemannian manifold, G a finite group, and H and K almost conjugate subgroups of G . If there is a surjective homomorphism from $\pi_1(M_0)$ onto G , then the finite covering spaces M_H and M_K of M_0 corresponding to the subgroups H and K , respectively, are iso-length spectral.*

When H and K are not conjugate in G , the manifolds M_H and M_K can often be shown to be nonisometric. For example, when M_0 is a surface, a generic hyperbolic metric on M_0 will produce nonisometric M_H and M_K ; see [4, Ch.12.7].

For surfaces, the simple closed geodesics often carry more topological information. Accordingly, the **simple length spectrum** $L^s(M)$ of M is defined to be the set of all lengths of simple closed geodesics on M counted with multiplicities; see [8]. Two manifolds M_1 and M_2 are said to be **simple iso-length spectral** if $L^s(M_1) = L^s(M_2)$.

Question 1. *Are there nonisometric simple iso-length spectral hyperbolic surfaces?*

In [8], McShane and Parlier give example of pairs of 4-holed spheres with geodesic boundary which have the same *interior simple length spectrum* (one ignores the boundary lengths). They do in fact have different boundary lengths, and so they have different simple length spectrum.

One can ask if Sunada's construction provides a positive resolution to Question 1.

Question 2. *Does Sunada's construction, for a given homomorphism $\rho : \pi_1(M_0) \rightarrow G$, generically give simple iso-length spectral surfaces?*

To answer Question 2, we choose one of the examples of almost conjugate subgroups Sunada provided in his paper [10].

Example. $G = (\mathbb{Z}/8\mathbb{Z})^\times \ltimes \mathbb{Z}/8\mathbb{Z}$ with usual action of $(\mathbb{Z}/8\mathbb{Z})^\times$ on $\mathbb{Z}/8\mathbb{Z}$.

$H = \{(1, 0), (3, 0), (5, 0), (7, 0)\}$ and $K = \{(1, 0), (3, 4), (5, 4), (7, 0)\}$ are almost conjugate but not conjugate.

Our main theorem is the following.

Theorem 1.1. *Let M_0 be a closed oriented surface of genus 2, G , H , and K the groups provided in the example above.*

There is a surjective homomorphism $\rho : \pi_1(M_0) \rightarrow G$ such that, for almost every $[m] \in \mathcal{T}(M_0)$, the corresponding iso-length spectral surfaces M_H and M_K are not simple iso-length spectral.

In fact, we prove a little bit more. We define the **length set** and the **simple length set** of a manifold M to be the set of all lengths of closed geodesics on M without multiplicities and the set of all lengths of simple closed geodesics on M without multiplicities, respectively. Then from the proof of Theorem 1.1 we have the following corollary.

Corollary 1.1. *The surfaces M_H and M_K in Theorem 1.1 have the same length set but they do not have the same simple length set.*

This corollary shows that the construction of length equivalent manifolds in [6] does not necessarily give simple length equivalent manifolds.

Outline of the paper. Section 2 contains the relevant background. In Section 3, we give the proof of the main theorem. The sketch of the proof is as follow. We begin by defining a surjective homomorphism $\rho : \pi_1(M_0) \rightarrow G$ and a closed curve α in M_0 . By Sunada's construction, the covering spaces $\pi_H : M_H \rightarrow M_0$ and $\pi_K : M_K \rightarrow M_0$ corresponding to the subgroups H and K are iso-length spectral. We then show that, for almost every $[m] \in \mathcal{T}(M_0)$, the induced metrics on M_H and M_K have the following property. In each of these two covering spaces M_H and M_K , there are exactly four closed geodesics having the same length as α , namely the two degree-one components of $\pi_H^{-1}(\alpha)$ (and $\pi_K^{-1}(\alpha)$) and their images under the lifts of the hyperelliptic involution $\tau : M_0 \rightarrow M_0$. We also show that these four closed geodesics on M_H are nonsimple while the other four closed geodesics on M_K are simple. Therefore M_H and M_K are not simple iso-length spectral.

We remark on one subtlety of the proof. According to [9], there are curves γ, γ' on M_0 such that for every hyperbolic metric m on M_0 , $\text{length}_m(\gamma) = \text{length}_m(\gamma')$. Although these are nonsimple on M_0 , they become simple in a finite sheeted cover, so must be accounted for in our proof.

2 Background

Let M be a closed oriented surface of genus $g \geq 2$. We denote the Teichmüller space of M by

$$\mathcal{T}(M) = \{[m] \mid m \text{ is a hyperbolic metric on } M\},$$

where $[m]$ represents the equivalence class via the equivalence relation $m \sim m'$ if there exists an isometry $f : (M, m) \rightarrow (M, m')$ such that $f \simeq id_M$, see e.g. [4].

Given $[m] \in \mathcal{T}(M)$, the holonomy homomorphism

$$\rho_m : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{R})$$

is well defined up to conjugation in $\mathrm{PSL}_2(\mathbb{R})$. This determines an embedding

$$\mathcal{T}(M) \rightarrow \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{R}))/\text{conjugation} \quad (1)$$

by $[m] \mapsto [\rho_m]$.

Let γ be an essential closed curve on M . The length function of γ

$$\mathrm{length}_{(\cdot)}(\gamma) : \mathcal{T}(M) \rightarrow \mathbb{R}_+$$

is defined as the length of the m -geodesic homotopic to γ . Using the holonomy homomorphism, one can compute

$$\mathrm{length}_{[m]}(\gamma) = 2 \cosh^{-1} \left(\frac{|\mathrm{tr}(\rho_m(\gamma))|}{2} \right). \quad (2)$$

The embedding (1) makes $\mathcal{T}(M)$ into a real analytic manifold. By (2), the length functions are analytic (see e.g. [5] or [1]). Since $\mathcal{T}(M)$ is connected, we then have the following theorem; see [8].

Theorem 2.1. *Let $c \in \mathbb{R}$, α and β be closed curves on M . The function*

$$f = c \cdot \mathrm{length}_{(\cdot)}(\beta) - \mathrm{length}_{(\cdot)}(\alpha) : \mathcal{T}(M) \rightarrow \mathbb{R}$$

is real analytic, in particular, $f \neq 0$ almost everywhere or $f = 0$ everywhere.

Let γ and γ' be closed curves on M . The geometric intersection number of γ and γ' is defined by

$$i(\gamma, \gamma') = \min_{\bar{\gamma}, \bar{\gamma}'} |(\bar{\gamma} \times \bar{\gamma}')^{-1}(\Delta)|,$$

where $\bar{\gamma}$ and $\bar{\gamma}'$ are in the homotopy classes $[\gamma]$ and $[\gamma']$, respectively, $\bar{\gamma} \times \bar{\gamma}' : S^1 \times S^1 \rightarrow M \times M$, and $\Delta \subset M \times M$ is diagonal.

The next theorem provides a tool for dealing with the phenomenon arising from [9].

Theorem 2.2. *Let γ, γ' be closed curves on M and $k \in \mathbb{R}$.*

If $\mathrm{length}_m(\gamma) = k \cdot \mathrm{length}_m(\gamma')$, for all $[m] \in \mathcal{T}(M)$, then $i(\gamma, \alpha) = k \cdot i(\gamma', \alpha)$, for all simple closed curves α on M .

Proof. For $k = 1$, a proof can be found in [7], for example. The same idea works here, and we sketch it.

Given a simple closed curve α , there exists a sequence $\{[m_n]\} \subset \mathcal{T}(M)$ such that

$$\frac{1}{n} \cdot \text{length}_{[m_n]}(\eta) \rightarrow i(\eta, \alpha),$$

for all closed curves η on M .

Now suppose $\text{length}_{[m]}(\gamma) = k \cdot \text{length}_{[m]}(\gamma')$ for all $[m] \in \mathcal{T}(M)$. Then

$$\frac{1}{n} \cdot \text{length}_{[m_n]}(\gamma) \rightarrow i(\gamma, \alpha)$$

and

$$\frac{k}{n} \cdot \text{length}_{[m_n]}(\gamma') \rightarrow k \cdot i(\gamma', \alpha).$$

So $k \cdot i(\gamma', \alpha) = i(\gamma, \alpha)$. □

The following theorem is shown in [7].

Theorem 2.3. *Given γ and γ' closed curves on M , if*

$$\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\gamma'),$$

for all $[m] \in \mathcal{T}(M)$, then $[\gamma] = \pm[\gamma']$ in $H_1(M)$.

3 Proof of the main theorem

Let M_0 be a closed oriented surface of genus 2. We write the fundamental group of M_0 as $\pi_1(M_0) = \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle$, see Figure 1.

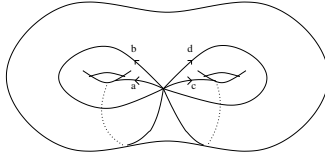


Figure 1: M_0 with the generators of $\pi_1(M_0)$.

Let G , H and K be groups given in the example in Section 1. We define a surjective homomorphism $\rho : \pi_1(M_0) \rightarrow G$ by

$$\rho(a) = (3, 0), \quad \rho(b) = (5, 0), \quad \rho(c) = (1, 0), \quad \text{and} \quad \rho(d) = (1, 1).$$

Let $\pi : M \rightarrow M_0$, $\pi_H : M_H \rightarrow M_0$ and $\pi_K : M_K \rightarrow M_0$ be the covering spaces of M_0 corresponding to $\ker(\rho)$, $\rho^{-1}(H)$ and $\rho^{-1}(K)$, respectively.

To help visualizing the covering space M , first we construct the covering space $\pi : M_N \rightarrow M_0$ corresponding to the subgroup $N = \mathbb{Z}/8\mathbb{Z}$ of G , as shown in Figure 2. Then we construct M from the surjective homomorphism $\sigma : \pi_1(M_N) \rightarrow N$, the restriction of ρ to $\pi_1(M_N) < \pi_1(M_0)$, see Figure 3. Observe that the generator of $\mathbb{Z}/8\mathbb{Z} \cong N < G$ translates each piece in Figure 3 to the right, and sends the last piece to the first piece.

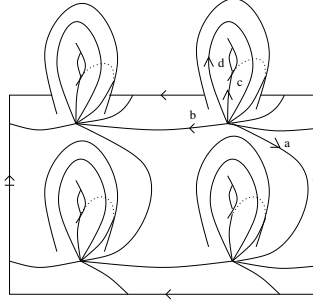


Figure 2: The covering space M_N .

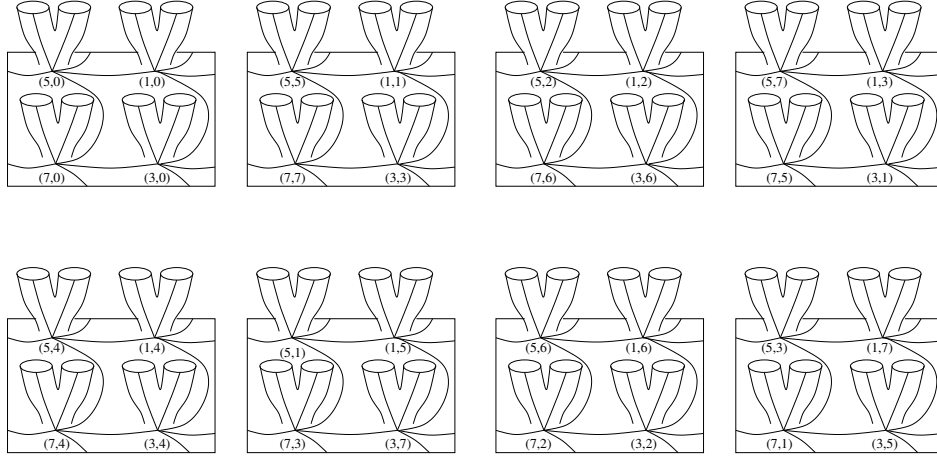


Figure 3: The covering space M .

Lemma 3.1. *Let $\alpha = abd[d, c^{-1}]d^{-1}$ be a closed curve on M_0 . Then $\pi_H^{-1}(\alpha) = \beta_1^H \cup \dots \cup \beta_5^H$, $\pi_K^{-1}(\alpha) = \beta_1^K \cup \dots \cup \beta_5^K$ where $\pi_H|_{\beta_i^H}$, $\pi_K|_{\beta_i^K}$ are degree one, for $i = 1, 2$, and degree two, for $i = 3, 4, 5$. Furthermore β_1^H , β_2^H are nonsimple and β_1^K , β_2^K are simple.*

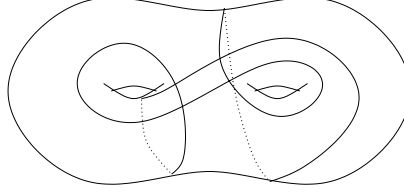


Figure 4: The closed curve α on M_0 .

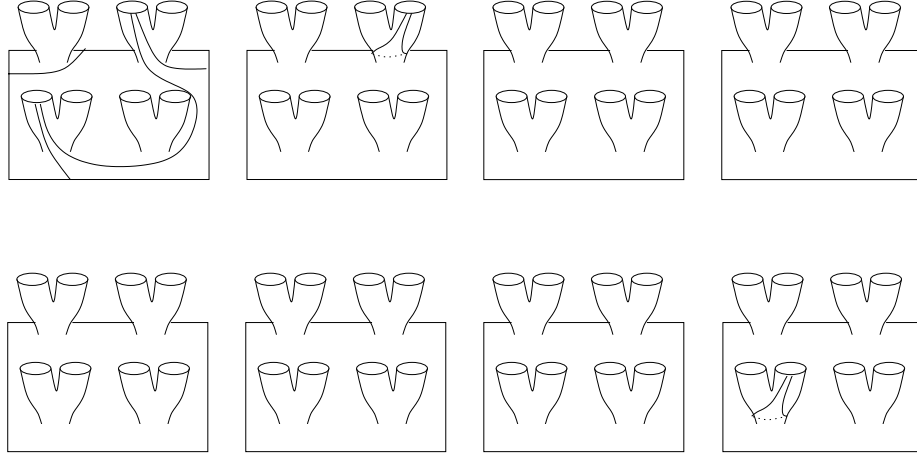


Figure 5: The covering space M and a component γ_1 of $\pi^{-1}(\alpha)$.

Proof. First we look at a component γ_1 of $\pi^{-1}(\alpha)$ in M , see Figure 5. Observe that the preimage of α is sixteen simple closed curves on M denotes $X = \{\gamma_1, \dots, \gamma_{16}\}$. G acts on X and this action is equivalent to the action of G on the cosets of $L = \text{Stab}_G(\gamma_1) = \{(1, 0), (7, 0)\}$. More precisely, the bijection

$$G//L \rightarrow X$$

given by

$$gL \mapsto g \cdot \gamma_1$$

is equivariant with respect to the actions of G . We assume $\{\gamma_1, \dots, \gamma_{16}\}$ are numbered so that

$$\begin{aligned} \gamma_1 &\rightarrow L, & \gamma_2 &\rightarrow (1, 1)L, & \gamma_3 &\rightarrow (1, 2)L, & \gamma_4 &\rightarrow (1, 3)L, \\ \gamma_5 &\rightarrow (1, 4)L, & \gamma_6 &\rightarrow (1, 5)L, & \gamma_7 &\rightarrow (1, 6)L, & \gamma_8 &\rightarrow (1, 7)L, \\ \gamma_9 &\rightarrow (3, 0)L, & \gamma_{10} &\rightarrow (3, 3)L, & \gamma_{11} &\rightarrow (3, 6)L, & \gamma_{12} &\rightarrow (3, 1)L, \\ \gamma_{13} &\rightarrow (3, 4)L, & \gamma_{14} &\rightarrow (3, 7)L, & \gamma_{15} &\rightarrow (3, 2)L, & \gamma_{16} &\rightarrow (3, 5)L. \end{aligned}$$

We use the above representations to compute H and K orbits under the actions of H and K on X . Then the H orbits partition $\{\gamma_1, \dots, \gamma_{16}\}$ as

$$\{\gamma_1, \gamma_9\}, \{\gamma_5, \gamma_{13}\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{16}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{12}, \gamma_{14}\}$$

and the K orbits partition $\{\gamma_1, \dots, \gamma_{16}\}$ as

$$\{\gamma_1, \gamma_{13}\}, \{\gamma_5, \gamma_9\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{14}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{10}, \gamma_{16}\}.$$

All closed curves in each H orbit lie above exactly one closed curve on M_H and all closed curves in each K orbit lie above exactly one closed curve on M_K . So we can write $\pi_H^{-1}(\alpha) = \beta_1^H \cup \dots \cup \beta_5^H$ and $\pi_K^{-1}(\alpha) = \beta_1^K \cup \dots \cup \beta_5^K$. We may associate $\beta_1^H, \beta_2^H, \beta_1^K$ and β_2^K with the orbits $\{\gamma_1, \gamma_9\}, \{\gamma_5, \gamma_{13}\}, \{\gamma_1, \gamma_{13}\}$ and $\{\gamma_5, \gamma_9\}$, respectively.

Next we observe that $\pi_H|_{\beta_i^H}, \pi_K|_{\beta_i^K}$ are degree one, for $i = 1, 2$, and degree two, for $i = 3, 4, 5$.

For the simplicity of $\beta_1^H, \beta_2^H, \beta_1^K$ and β_2^K , we look at their associated orbits. We observe that γ_1 intersects $\gamma_9 = (3, 0) \cdot \gamma_1$ nontrivially by inspecting Figure 3 for the actions of G and Figure 5 for the picture of γ_1 . Similarly we can compute

$$\begin{aligned} \gamma_1 \cap \gamma_9 &\neq \emptyset, & \gamma_5 \cap \gamma_{13} &\neq \emptyset, \\ \gamma_1 \cap \gamma_{13} &= \emptyset, & \gamma_5 \cap \gamma_9 &= \emptyset. \end{aligned}$$

Since the H orbit $\{\gamma_1, \gamma_9\}$ corresponding to β_1^H contains intersecting curves, β_1^H is nonsimple. Similarly, β_2^H is also nonsimple. Since the K orbit $\{\gamma_1, \gamma_{13}\}$ corresponding to β_1^K contains pairwise disjoint curves, β_1^K is simple. Similarly, β_2^K is also simple. \square

To prove Theorem 1.1, we will show that generically a hyperbolic metric on M_0 lifted to a hyperbolic metric on M_H has the property that there are

exactly four closed curves on M_H having the same length as β_1^H (and β_2^H) and these four closed curves are nonsimple. In the previous Lemma, we found two such closed curves, namely β_1^H and β_2^H . Lemma 3.2 provides the other two closed curves and we will use Lemma 3.3 to show that there are exactly four such closed curves. Since M_K has a simple closed curve, β_1^K , of the same length in its lifted metric, M_H and M_K cannot be simple iso-length spectral.

Let $\tau : M_0 \rightarrow M_0$ be the hyperelliptic involution. τ is isotopic to an isometry for any hyperbolic metric on M_0 . So for any curve λ on M_0 , $\text{length}_{M_0}(\lambda) = \text{length}_{M_0}(\tau(\lambda))$. For a specific basepoint, the induced map $\tau_* : \pi_1(M_0) \rightarrow \pi_1(M_0)$ can be computed to be

$$\begin{aligned} \tau_*(a) &= a^{-1}, & \tau_*(b) &= b^{-1}, \\ \tau_*(c) &= ac^{-1}dc^{-1}d^{-1}ca^{-1}, & \tau_*(d) &= b^{-1}ad^{-1}ba^{-1}. \end{aligned}$$

We have the following lemma.

Lemma 3.2. *The hyperelliptic involution $\tau : M_0 \rightarrow M_0$ lifts to $\tau_H : M_H \rightarrow M_H$ and $\tau_K : M_K \rightarrow M_K$. In particular, $\tau_H(\beta_i^H) \subset M_H$ is nonsimple and $\tau_K(\beta_i^K) \subset M_K$ is simple, for $i = 1, 2$.*

Proof. Let $\psi : G \rightarrow G$ be the automorphism of G defined by $\psi(j, k) = (j, -k)$, for any element $(j, k) \in G$. Then we can compute $\psi \circ \rho = \rho \circ \tau_*$ and $H = \psi^{-1}(H)$. So $\rho^{-1}(H) = \rho^{-1}(\psi^{-1}(H)) = \tau_*^{-1}(\rho^{-1}(H))$. Thus

$$\tau_*((\pi_H)_*(\pi_1(M_H))) = \tau_*(\rho^{-1}(H)) = \rho^{-1}(H) = (\pi_H)_*(\pi_1(M_H)).$$

Hence the lifting criterion implies that we may lift τ to τ_H . The existence of a lift τ_K to M_K is proven in the same way. \square

Lemma 3.3. *For almost every $[m] \in \mathcal{T}(M_0)$, if γ is a closed curve, $k \in \mathbb{Q}$ and*

$$k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$$

then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$.

Proof. For any γ and any k , either $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ is true for every $[m]$ or $k \cdot \text{length}_{[m]}(\gamma) \neq \text{length}_{[m]}(\alpha)$ for almost every $[m]$, by Theorem 2.1. So it suffices to show that if $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, for every $[m]$, then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$.

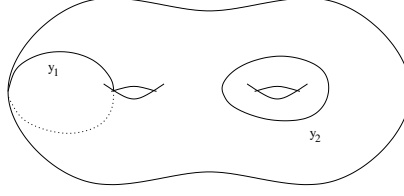


Figure 6: The simple closed curves x_1 and x_2 on the surface M_0 .

Let y_1 be a simple closed curve as shown in Figure 6. The geometric intersection number of α and y_1 is $i(\alpha, y_1) = 1$. Since $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, by Theorem 2.2, $k \cdot i(\gamma, y_1) = i(\alpha, y_1) = 1$. Since the geometric intersection numbers are nonnegative integers, $k = 1$. To prove that $\gamma = \alpha$ or $\tau(\alpha)$, we find some necessary conditions for γ to have the same length as α , for every $[m] \in \mathcal{T}(M_0)$

Let y_2 be the simple closed curve shown in Figure 6. Since $i(\gamma, y_2) = i(\alpha, y_2) = 0$ by Theorem 2.2, γ and α are contained in $M_0 - y_2$.

We cut M_0 along the simple closed curve y_2 to get a torus with two holes and change the basis $\{a, b, d\}$ to the basis $\{a, b, x = da^{-1}\}$, see Figure 7. Then $\alpha = abxaba^{-1}b^{-1}x^{-1}$ and $\tau_*(\alpha) = a^{-1}b^{-1}b^{-1}x^{-1}ba^{-1}b^{-1}axb$. Consider

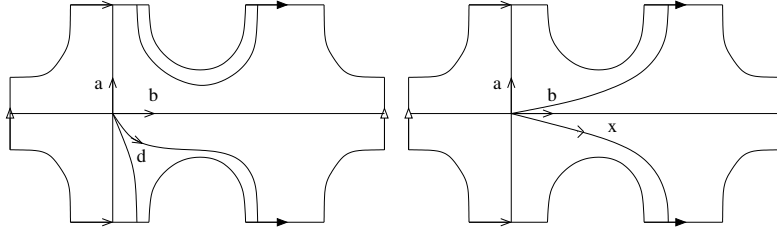


Figure 7: The torus with two holes, $M_0 - x_2$.

the spine as shown in Figure 8, we homotope α and γ into spine, as edge loops without backtracking. Then by considering metrics on M_0 where length of some of the edges are bounded and others tend to infinity, we see that in order for γ to have the same length as α in M_0 ,

$$\begin{aligned} \# \{a_1 \text{ edges of } \gamma\} &= \# \{a_1 \text{ edges of } \alpha\} = 3, \\ \# \{x_1 \text{ edges of } \gamma\} &= \# \{x_1 \text{ edges of } \alpha\} = 3, \\ \# \{b_1 \text{ edges of } \gamma\} + \# \{b_2 \text{ edges of } \gamma\} &= \# \{b_1 \text{ edges of } \alpha\} + \# \{b_2 \text{ edges of } \alpha\} = 8. \end{aligned}$$

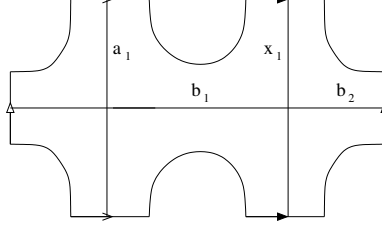


Figure 8: The torus with two holes, $M_0 - x_2$ with spine.

Since $\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ and $[\alpha] = [ab] \in H_1(M_0)$, $[\gamma] = \pm[ab] \in H_1(M_0)$, by Theorem 2.3. Thus from the observation of the edge counts above (replacing γ with γ^{-1} if necessary), we have the following conditions;

1. γ consists of exactly two a 's, one a^{-1} , one x , and one x^{-1} ,
2. $\# \{b^{-1}\text{'s in } \gamma\} = \# \{b\text{'s in } \gamma\} - 1$, and
3. $\# \{b_1 \text{ edges of } \gamma\} + \# \{b_2 \text{ edges of } \gamma\} = 8$.

Next we find all closed curves on M_0 satisfying these three conditions. By the conditions above we know the exact number of a 's, a^{-1} 's, x 's, and x^{-1} that appear in γ . So we only need to determine the possible number of b 's and b^{-1} . To do this, we note that while the number a_1 -edge and the number of x_1 -edge can be computed directly by counting the number of $\{a, a^{-1}\}$ and $\{x, x^{-1}\}$, respectively, some combinations of x 's and b 's provide cancellations in the sum of b_1 and b_2 -edge count. One example is that x alone contributes 2 to the sum of b_1 and b_2 -edge count, b alone also contributes 2 to the sum of b_1 and b_2 -edge count but xb contributes only 2 to the sum of b_1 and b_2 -edge count.

Taking this type of cancellation into consideration, we can produce a list A of 4320 words in $\{a^{\pm 1}, b^{\pm 1}, x^{\pm 1}\}$ that contains all curves satisfying the three conditions.

One can explicitly construct $[m] \in \mathcal{T}(M_0)$, a hyperbolic metric on M_0 such that

$$\rho_m(a) = \begin{pmatrix} 5/3 & 3/4 \\ 3/4 & 5/4 \end{pmatrix},$$

$$\rho_m(b) = \begin{pmatrix} 4 & 0 \\ 0 & 1/4 \end{pmatrix},$$

$$\rho_m(x) = \begin{pmatrix} 5/3 & -16/3 \\ -1/3 & 5/3 \end{pmatrix}.$$

Then the trace of $\rho_m(\alpha)$ is

$$\text{tr}(\rho_m(\alpha)) = 109505/2048.$$

By using Mathematica, we have that the elements in A having the same trace squared as α are α and $\tau(\alpha)^{-1}$.

So, by equation (2), the only curves in A that have the same length in M_0 as α are α and $\tau(\alpha)$.

Thus if $\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, for every $[m] \in \mathcal{T}(M_0)$, then $\gamma = \alpha$ or $\tau(\alpha)$. \square

Proof of Theorem 1.1. Let $\rho : \pi_1(M_0) \rightarrow G$ be the surjective homomorphism defined in this section.

Let $\alpha = abd[d, c^{-1}]d^{-1}$ be a closed geodesic on M_0 .

By Lemma 3.1 and Lemma 3.2, for almost every $[m] \in \mathcal{T}(M_0)$, there are four nonsimple closed geodesics $\{\beta_1^H, \beta_2^H, \tau_H(\beta_1^H), \tau_H(\beta_2^H)\}$ on M_H having length $l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha)$ and there are four simple closed geodesics $\{\beta_1^K, \beta_2^K, \tau_K(\beta_1^K), \tau_K(\beta_2^K)\}$ on M_K having length l .

If γ^H is a closed geodesic on M_H having length

$$l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha),$$

then $\pi_H(\gamma^H)$ is a closed geodesic on M_0 having length

$$k \cdot l = k \cdot \text{length}_{[m]}(\beta_1^H) = k \cdot \text{length}_{[m]}(\gamma),$$

for some $k = 1, 1/2, 1/4$, or $1/8$, since the degree of π_H and π_K is 8.

By Lemma 3.3, $k = 1$ and $\pi_H(\gamma^H) = \alpha$ or $\tau(\alpha)$. Thus γ^H is one of the four nonsimple closed curves above. Hence there are exactly four closed curves on M_H having length l and those four closed curves are nonsimple. Similarly, there are exactly four closed curves on M_K having length l and those four closed curves are simple.

Therefore M_H and M_K are not simple iso-length spectral. \square

Proof of Corollary 1.1. As the proof of Theorem 1.1 shows, for almost every $[m] \in \mathcal{T}(M_0)$, there is a simple closed geodesic on M_K with the same length as α on M_0 , but no such simple geodesic on M_H . Therefore, M_H and M_K are not simple length equivalent. \square

4 Final discussion

Theorem 1.1 should hold for any surjective homomorphism $\rho : \pi_1(M_0) \rightarrow G$ and for any closed surface M_0 . Indeed, it can be shown that for G as in Theorem 1.1 and any ρ , there is a genus 2 or 3 subsurface $\Sigma \subset M_0$ so that the restriction $\rho|_{\pi_1(\Sigma)}$ is surjective. Then, one can list all such surjective homomorphisms and try to construct a curve α in Σ playing the role of α in the proof of Theorem 1.1. This does not seem to provide much new information, and even for the cases analyzed by the author, the resulting presentation is significantly more complicated. It would be interesting to find an approach that works for all homomorphisms simultaneously.

Another class of examples that would be interesting to analyze with respect to Question 2 are those given in [2] and [3], as the proof that the surfaces are iso-length spectral is more directly geometric.

Acknowledgements

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References

- [1] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.
- [2] Robert Brooks and Richard Tse. Isospectral surfaces of small genus. *Nagoya Math. J.*, 107:13–24, 1987.
- [3] Peter Buser. Isospectral Riemann surfaces. *Ann. Inst. Fourier (Grenoble)*, 36(2):167–192, 1986.
- [4] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.
- [5] Steven Kerckhoff. Earthquakes are analytic. *Commentarii Mathematici Helvetici*, 60:17–30, 1985. 10.1007/BF02567397.

- [6] C. J. Leininger, D. B. McReynolds, W. D. Neumann, and A. W. Reid. Length and eigenvalue equivalence. *Int. Math. Res. Not. IMRN*, (24):Art. ID rnm135, 24, 2007.
- [7] Christopher J. Leininger. Equivalent curves in surfaces. *Geom. Dedicata*, 102:151–177, 2003.
- [8] Greg McShane and Hugo Parlier. Multiplicities of simple closed geodesics and hypersurfaces in Teichmüller space. *Geom. Topol.*, 12(4):1883–1919, 2008.
- [9] Burton Randol. The length spectrum of a Riemann surface is always of unbounded multiplicity. *Proc. Amer. Math. Soc.*, 78(3):455–456, 1980.
- [10] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Ann. of Math. (2)*, 121(1):169–186, 1985.

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